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exceeds a right angle by less than the angle *DAC*. But then angle *DAC*, or *DAM*, together with angle *AMD* will be greater than a right angle. Wherefore, the obtuse angle *ADM* being added, the three angles together of the triangle *ADM* will be greater than two right angles, which is against the hypothesis of acute angle.

Therefore every straight *AC*, which cuts that angle *BAX*, finally at a finite or terminated distance (hypothesis of acute angle) must meet *BX* in a certain point *P*. Quod etc.

Corollary I. Hence no straight *AZ*, which toward the parts of the points *X* makes an acute angle greater than *BAX* can ever meet *BX*, either at a finite, or at an infinite distance. For as far as so should happen, now *AX* dividing angle *BAZ*, ought (against the premised supposition) to meet *BX* at a finite distance, as this is demonstrated of the straight *AC* dividing angle *BAX*.

Corollary II. Moreover it follows that no determinate acute angle will be the maximum of all under which a straight line produced from point *A* meets *BX* at finite distance. For if toward the parts of the point *X* you assume any point higher than the point *P*, it follows that the straight joining point *A* with this higher point will make with *AB* a greater angle than angle *BAP*. And so ever without any intrinsic end. Wherefore angle *BAX* (since indeed *AX* both always approaches to *BX*, and meets it only at an infinite distance) will be the outside limit of all acute angles under which straights produced from that point *A* meet the aforesaid *BX* at a finite distance.

[To be Continued.]

SOPHUS LIE'S TRANSFORMATION GROUPS.

A SERIES OF ELEMENTARY, EXPOSITORY ARTICLES.

By **EDGAR ODELL LOVETT**, Princeton University.

II.

The Group of One Parameter. The Infinitesimal Transformation. Existence of an Infinitesimal Transformation in a Group of One Parameter.

5. Consider the plane as a point manifoldness, *i. e*, a space whose space element is the point. The plane will then be two-dimensional,* *i. e* contain ∞^2

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*This idea and its bearing in the paragraph are emphasized here not to introduce any unnecessary ultra refinement but because of their use in geometrical illustrations to appear in succeeding articles. For example, the plane is one, two, three, or four dimensional according as, a circle with fixed center, the straight line, a circle of general position, or parabola, be taken as space element, since there are ∞^1, ∞^2, ∞^3, ∞^4, of these elements respectively in the plane. Similarly if the straight line is element it has no dimension, the point has one dimension, the plane two dimensions, and ordinary space four dimensions.

elements, or in other words the position of a point in the plane will be determined by two parameters, the coördinates of the point.

A point-transformation of the plane into itself is an operation by which every point in the plane is conveyed into the position of some point in the same plane. In order to represent this operation analytically, let us take as the coördinate system of reference an ordinary rectangular Cartesian system, x, y ; then the point transformation is expressed by two equations of the form

$$x_1=\varphi(x, y), \quad y_1=\psi(x, y), \qquad (1)$$

where (x, y) is the original point and (x_1, y_1) the transformed point. It is further assumed that the transformation is of such a nature that every point (x_1, y_1) of the plane may be regarded as having originated from some point in the plane by effecting the transformation. This geometrical assumptions finds its analytical condition in the demand that the two functions $\varphi(x, y)$ and $\psi(x, y)$ be independent functions and thus the preceding equations are soluble theoretically with regard to x and y. For, suppose that φ and ψ were not independent, and for example let

$$\varphi=n\alpha(x, y), \quad \psi=m\alpha(x, y) ;$$

then eliminating $\alpha(x, y)$ from the equations of the transformation

$$x_1=n\alpha(x, y), \quad y_1=m\alpha(x, y)$$

we find that the points (x, y) of the plane are transformed into the points (x_1, y_1) of the straight line

$$y_1=\frac{m}{n}x_1,$$

and hence point of general position no longer is conveyed into point of general position by the transformation.

If the equations (1) be solved with regard to the variables x, y, there result two equations of the form

$$x=\overline{\varphi}(x_1, y_1), \quad y=\overline{\psi}(x_1, y) \qquad (2)$$

which represent a transformation that carries the point (x_1, y_1) back into the position (x, y) ; this transformation (2) is called the *inverse* of the transformation (1). If the transformation (1) be followed by the transformation (2), that is, if the two transformations be carried out successively, we have the two equations

$$x_1=x, \quad y_1=y.$$

These equations are very particular cases of equations (1) and hence should represent a transformation. The transformation which they represent obviously transforms a point into itself, or in other words, it leaves all points at rest, for this reason it is called the *identical* transformation.

6. If the equations of a transformation

$$x_1=\varphi(x,\ y,\ a),\quad y_1=\psi(x,\ y,\ a) \tag{3}$$

contain an arbitrary constant a, these equations no longer represent a single transformation but a family of ∞^1 transformations, since the arbitrary parameter a may assume all values from $-\infty$ to $+\infty$. Let us make the hypothesis that the equations (3) represent such a family of transformations that *the successive application of any two transformations of the family is equivalent to a transformation belonging to the same family ;* in this case the family (3) is called a *group ;* since the group contains *one* parameter a and hence ∞^1 transformations, the group is called a *one parameter group*, or a *group* of one parameter, or symbolically a G_1. Further, since the parameter varies continuously the group is said to be a *continuous* group. As the group contains a finite number of parameters, in this case but one, namely a, it is a *finite continuous group.* The sentence above in italics expresses the group property of the family. A footnote in the preceding article calls attention to the fact that the group property is peculiar to certain classes of families and not common to all of them.

The analytical criterion for a one-parameter group as just defined reveals itself in the following manner. The transformation

$$T_1 \qquad x_1=\varphi(x,\ y,\ a),\quad y_1=\psi(x,\ y,\ a), \tag{4}$$

changes the point $(x,\ y)$ into the point $(x_1,\ y_1)$; let T_1 be followed by the transformation T_2 which corresponds to the value a_1 of the parameter and changes the point $(x_1,\ y_1)$ into the point $(x_2,\ y_2)$, given by the equations

$$T_2 \qquad x_2=\varphi(x_1,\ y_1,\ a_1),\quad y_2=\psi(x_1,\ y_1,\ a_1). \tag{5}$$

The transformation, T_3, say, which will carry the point $(x,\ y)$ directly into the position $(x_2,\ y_2)$ is found by eliminating $(x_1,\ y_1)$ from the equations (4) and (5). The elimination yields

$$T_3\equiv T_1T_2\ \begin{cases} x_2=\varphi\{\varphi(x,\ y,\ a),\ \psi(x,\ y,\ a),\ a_1\}, \\ y_2=\psi\{\varphi(x,\ y,\ a),\ \psi(x,\ y,\ a),\ a_1\}. \end{cases} \tag{6}$$

If this transformation is to belong to the original family it must be capable of expression in the form

$$x_2=\varphi(x,\ y,\ \lambda),\quad y_2=\psi(x,\ y,\ \lambda),$$

where λ is a certain function of a and a_1 alone.

Hence the criterion sought is that the two equations

$$\varphi\{\varphi(x,\ y,\ a),\ \psi(x,\ y,\ a),\ a_1\}\equiv\varphi\{x,\ y,\ \lambda(a,\ a_1)\},$$

$$\psi\{\varphi(x, y, a), \psi(x, y, a), a_1\} \equiv \psi\{x, y, \lambda(a, a_1)\},$$

must exist identically for all values of x, y, a, and a_1.

7. In the sequel we shall study only those continuous groups* which contain the *inverse* transformation of every transformation of the group, *i. e.* to a tranformation corresponding to the parameter a,

$$T_1 \qquad x_1 = \varphi(x, y, a), \quad y_1 = \psi(x, y, a),$$

there corresponds a transformation of the family whose parameter is $\bar{a}$ say,

$$T_2 \qquad x_2 = \varphi(x_1, y_1, \bar{a}), \quad y_2 = \psi(x_1, y_1, \bar{a}),$$

such that T_2 cancels T_1 and gives

$$x_2 = x, \quad y_2 = y,$$

the identical transformation.

Accordingly, if the transformations of a group are inverse in pairs the group contains the *identical* transformation. Let a_0 be the value of the parameter which gives the identical transformation, then

$$\varphi(x, y, a_0) \equiv x, \quad \psi(x, y, a_0) \equiv y.$$

A transformation of the family whose parameter is $a_0 + \delta a$, where δa is an indefinitely small quantity will move the point (x, y) through only an infinitesimal distance, such a transformation is called an *infinitesimal transformation*, where by an *infinitesimal* transformation of the group is meant a transformation whose parameter differs by an infinitesimal from that value of the parameter which gives the *identical transformation.*

8. There is a most intimate connection between the notions *infinitesimal transformation* and *one parameter group.* It is proposed to derive now three fundamental theorems of LIE which establish this relationship. The first proves that *every one parameter group contains an infinitesimal transformation*, the second that *every infinitesimal transformation generates a one parameter group*, and the third that *a one parameter group contains but one infinitesimal transformation.*

The three theorems show that an infinitesimal transformation may be taken as the representative of a one parameter group.

That a group of one parameter contains an infinitesimal transformation may be seen geometrically in the following manner:

Let a transformation of the group which corresponds to the parameter α and which is designated for convenience by (α) carry the point p (x, y) to the position p_1, (x_1, y_1). By assumption the inverse of (α) is contained in the group. Let the parameter of this inverse transformation be $\bar{\alpha}$; $\bar{\alpha}$ is a certain function of α. The transformation $(\bar{\alpha})$ changes all points p_1 into the points p again respectively. A transformation whose parameter differs infinitesimally from $\bar{\alpha}$, say $\bar{\alpha}+\delta\alpha$, will carry the point p_1 not back to p, but to a position at an infinitesimal distance from p, say p'. The successive performance of (α) and $(\bar{\alpha}+\delta\alpha)$ will carry p to p_1 and then to p'; but (α) and $(\bar{\alpha}+\delta\alpha)$ belong to the group, hence the third transformation to which they are equivalent belongs to the group ; that is, the transformation which carries p to p', a point infinitesimally near, belongs to the group, or in other words the group contains an infinitesimal transformation.

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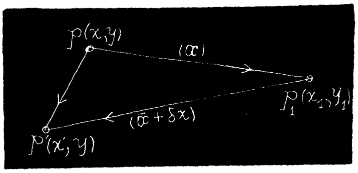
*The reader must be reminded that this limitation is really not a restriction. LIE has proved in volume III of the Theory of Transformation Groups, Theorem 26, that the defining equations of any continuous group can be derived from those of a group whose transformations are inverse in pairs.



This geometric process may now be clothed in analytic garb. The first transformation (α) is given by the equations

$$x_1=\varphi(x,\ y,\ \alpha),\quad y_1=\psi(x,\ y,\ \alpha)\ ; \tag{7}$$

the second transformation $(\bar{\alpha}+\delta\alpha)$ by

$$x'=\varphi(x_1,\ y_1,\ \bar{\alpha}+\delta\alpha),\quad y'=\psi(x_1,\ y_1,\ \bar{\alpha}+\delta\alpha). \tag{8}$$

The elimination of x_1, y_1 from these equations gives the transformation which carries p to p', namely

$$x'=\varphi\{\varphi(x,\ y,\ \alpha),\ \psi(x,\ y,\ \alpha),\ \bar{\alpha}+\delta\alpha\},\ y'=\psi\{\varphi(x,\ y,\ \alpha),\ \psi(x,\ y,\ \alpha),\ \bar{\alpha}+\delta\alpha\}. \tag{9}$$

Developing* these values in powers of $\delta\alpha$ we have

$$\left.\begin{aligned} x'&=\varphi\{\varphi(x,\ y,\ \alpha),\ \psi(x,\ y,\ \alpha),\ \bar{\alpha}\}+\frac{\partial\varphi\{\varphi(x,\ y,\ \alpha),\ \psi(x,\ y,\ \alpha),\ \bar{\alpha}\}}{\partial\bar{\alpha}}\ \frac{\delta\alpha}{1}+\ldots\ldots,\\ y'&=\psi\{\varphi(x,y,\ \alpha),\ \psi(x,\ y,\ \alpha),\ \bar{\alpha}\}+\frac{\partial\psi\{\varphi(x,\ y,\ \alpha),\ \psi(x,\ y,\ \alpha),\ \bar{\alpha}\}}{\partial\bar{\alpha}}\ \frac{\delta\alpha}{1}+\ldots\ldots \end{aligned}\right\} \tag{10}$$

Now since the transformetions (α) and $(\bar{\alpha})$ are inverse

$$\varphi\{\varphi(x,\ y,\ \alpha),\ \psi(x,\ y,\ \alpha),\ \bar{\alpha}\}\equiv x,\quad \psi\{\varphi(x,\ y,\ \alpha),\ \psi(x,\ y,\ \alpha),\ \bar{\alpha}\}\equiv y\ ;$$

hence the equations of the transformation which changes p into p' are

$$\left.\begin{aligned} x'&=x+\frac{\partial\varphi\{\varphi(x,\ y,\ \alpha),\ \psi(x,\ y,\ \alpha)\ \bar{\alpha}\}}{\partial\bar{\alpha}}\ \frac{\delta\alpha}{1}+\ldots\ldots,\\ y'&=y+\frac{\partial\psi\{\varphi(x,\ y,\ \alpha),\ \psi(x,\ y,\ \alpha),\ \bar{\alpha}\}}{\partial\bar{\alpha}}\ \frac{\delta\alpha}{1}+\ldots\ldots, \end{aligned}\right\} \tag{11}$$

and in this form they represent an infinitesimal transformation since the values of x' and y' differ from x and y respectively by infinitely small quantities. It is easy to see that the coefficients of $\delta\alpha$ do not vanish, for if we put for $\varphi(x, y, \alpha)$ and $\psi(x, y, \alpha)$ their equals x_1 and y_1 respectively, these coefficients equated to zero are

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*It is to be remarked once for all that all functions here considered are regular analytic functions and hence expansible by Taylor's Theorem.

$$\frac{\partial\varphi(x_1, y_1, \bar{\alpha})}{\partial\bar{\alpha}}\equiv0, \quad \frac{\partial\psi(x_1, y_1, \bar{\alpha})}{\partial\bar{\alpha}}\equiv0.$$

But these last identities assert that φ and ψ are free from $\bar{\alpha}$, that is, in general the equations of the group contain no parameter which is contrary to hypothesis.

The quantity $\bar{\alpha}$ is a function of α, since to a transformation $(\bar{\alpha})$ there corresponds, by hypothesis, a completely determinate inverse transformation $(\bar{\alpha})$. The equations (1) of the infinitesimal transformation may be written in the form

$$x'=x+\xi(x, y, \alpha)\delta\alpha+\ldots. , \quad y'=y+\eta(x, y, \alpha)\delta\alpha+\ldots.*$$

LIE thus arrives at the following theorem :

I. *Every one parameter group whose transformations are inverse in pairs contains at least one infinitesimal transformation.*

Princeton University, 22 October, 1897.

[To be Continued.]

ARITHMETIC.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

83. Proposed by the late REV. G. W. BATES, A. M., Pastor of M. E. Church, Dresden City, Ohio.

A has three notes; the first and second, $1000 each, and the third $457; all dated April 1, 1884. The first is due April 1, 1888, second, April 1, 1889, and the third, April 1, 1890, and each bearing interest at 6%. What must B pay for the three notes September 21, 1886, that the investment will bring him 8% compound interest ?

Solution by G. B. M. ZERR, A. M., Ph. D., President of Russell College, Lebanon, Mo.

(I). Regarding the notes as bearing simple interest. We get

$1000 × 1.24=$1240, amount of first note.

$1000 × 1.30=$1300, amount of second note.

$457 × 1.36=$621.52, amount of third note.

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*These equations contain a constant α which can be arbitrarily chosen, hence we can find an infinitesimal transformation of the group in many different ways. But the sequel will show that all these, excepting a constant factor, are identical in their terms of the first order of infinitesimals.